Math and Implementation Notes for Credit Default Swap Pricing

A standard model follows the ISDA CDS Standard Model

Francis Zhang ©2023

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Notation

- $\Delta(s,t)$: year fraction between s and t. $\Delta(t_i)$ means $\Delta(t_{i-1},t_i)$.
- There are M payments on $t_1, t_2, \dots, t_i, \dots, t_M$
- Discount factor $D(t)$
- Survival probability $Q(t)$
- Coupon rate C
- Notional N
- Default time τ

The Premium Leg

There are two terms of the premium leg, one is the regular coupon payment, and the other is a single payment of the accrued premium in the event of default.

The regular coupon payments is just the risky discount of the future cash flow, which is,

$$PV_{\text{Premiums Only}} = NC\mathbb{E}\left[\sum_{i=1}^{M} \Delta\left(t_i\right) e^{-\int_0^{t_i} r(s)ds} \mathbb{I}_{t_i<\tau}\right] = NC\sum_{i=1}^{M} \Delta\left(t_i\right) D\left(t_i\right) Q\left(t_i\right)$$

The accrued part involves the integration, and is,

$$PV_{\text{Accrued Interest}} = NC\sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \Delta\left(t_{i-1}, s\right) D(s)(-dQ(s))$$

We could ignore the N and C in the above formula, and define the risky PV01 as,

$$\text{RPV01} = \sum_{i=1}^{M} \Delta\left(t_i\right) D\left(t_i\right) Q\left(t_i\right) + \sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \Delta\left(t_{i-1}, s\right) D(s)(-dQ(s))$$

Overall, we have,

$$PV_{\text{Premium}} = PV_{\text{Premiums only}} + PV_{\text{accrued interest}} = NC \times \text{RPV01}$$

Approximation 1: Halfway Default

To avoid the integration, we could assume that,

1. The default occurs halfway through the payment period, and the accrued premium is then $C\Delta(t_{i-1}, t_i)/2$. The probabilify of default during this payment period is the reduction of survival probability $Q(t_{i-1}) - Q(t_i)$.
2. The accrued interest is paid at the end of the payment period for simplicity.

That is, the integral term can be approximated by,

$$\int_{t_{i-1}}^{t_i} \Delta\left(t_{i-1}, s\right) D(s)(-dQ(s)) \simeq \frac{1}{2} \Delta\left(t_{i-1}, t_i\right) D\left(t_i\right) \left(Q\left(t_{i-1}\right) - Q\left(t_i\right)\right)$$

Combined with the regular coupon payment, we have,

$$\text{RPV01} = \frac{1}{2} \sum_{n=1}^{M} \Delta\left(t_{i-1}, t_i\right) D\left(t_i\right) \left(Q\left(t_{i-1}\right) + Q\left(t_i\right)\right)$$

Approximation 2: Constant Hazard and Interest Rate

Another approximation is to assume that the hazard rate and interest rate are constant, for some $t \in [t_{i-1}, t_i]$, we have,

$$D(t) = D(t_{i-1}) e^{-r(t - t_{i-1})}$$
$$Q(t) = Q(t_{i-1}) e^{-\lambda(t - t_{i-1})}$$

Then the integral term for the accrued interest becomes,

$$
\begin{aligned}
\int_{t_{i-1}}^{t_i} \Delta\left(t_{i-1}, s\right) D(s)(-dQ(s)) &= \int_{t_{i-1}}^{t_i} \Delta\left(t_{i-1}, s\right) D(t_{i-1}) e^{-r(s-t_{i-1})}(-dQ(t_{i-1}) e^{-\lambda(s-t_{i-1})}) \\
&= D(t_{i-1}) Q(t_{i-1}) \lambda \int_{t_{i-1}}^{t_i} (s - t_{i-1}) e^{-(r+\lambda)(s-t_{i-1})} ds \\
&= D(t_{i-1}) Q(t_{i-1}) \lambda \int_0^{t_i - t_{i-1}} s e^{-(r+\lambda)s} ds
\end{aligned}
$$

Denote $\alpha = r + \lambda$, and $\Delta = t_i - t_{i-1}$, we can simplify the integral the above as,

$$
\lambda e^{-\alpha t_{i-1}} \int_0^{t_i - t_{i-1}} s e^{-\alpha s} ds = \lambda e^{-\alpha t_{i-1}} \times \frac{1 - e^{-\alpha \Delta}(1 + \alpha \Delta)}{\alpha^2}
$$