X, Y uniform $(0,1)$. Find $f_Z(z)$ where $Z = X+Y$.

$$\mathbb{P}(X+Y \le z) = \int_0^1 \mathbb{P}(Y \le z-x \mid X=x) f_X(x)\,dx$$

$$= \int_0^1 \mathbb{P}(Y \le z-x)\,dx$$

① $z \in (0,1)$

$$= \int_0^z (z-x)\,dx$$

$$= \left(zx - \frac{1}{2}x^2\right)\Big|_{x=0}^{x=z} = \frac{1}{2}z^2$$

② $z \in (1,2)$

$$= \int_{z-1}^1 (z-x)\,dx + \int_0^{z-1} 1\,dx$$

$$= \left(zx - \frac{1}{2}x^2\right)\Big|_{x=z-1}^{x=1} + x\Big|_{x=0}^{x=z-1}$$

$$= z - \frac{1}{2} - z^2 + z + \frac{1}{2}(z^2 - 2z + 1) + z - 1$$

$$= -z^2 + 2z \cancel{-\frac{1}{2}} + \frac{1}{2}z^2 \cancel{-z} + \cancel{\frac{1}{2}} \cancel{+z} - 1$$

$$= -\frac{1}{2}z^2 + 2z - 1$$

$$f_Z(z) = \begin{cases} z, & \text{for } 0 < z < 1 \\ 2-z, & \text{for } 1 \le z < 2 \\ 0, & \text{otherwise} \end{cases}$$

X, Y uniform $(0,1)$, Find $f_Z(z)$ where $Z = XY$

$$\mathbb{P}(XY \le z) = \int_0^1 \mathbb{P}\left(Y \le \frac{z}{x} \,\middle|\, X = x\right) \cdot f_X(x)\, dx$$

$$= \int_0^1 \mathbb{P}\left(Y \le \frac{z}{x}\right) dx$$

$$= \int_0^z 1 \, dx + \int_z^1 \frac{z}{x}\, dx$$

$$= z + (z \ln x)\Big|_z^1$$

$$= z - z \ln z$$

$$f_Z(z) = 1 - (\ln z + 1) = -\ln z, \quad z \in [0,1].$$